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Central extensions and realizations of one-dimensional Galilean systems and quantization

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Abstract. The unitary irreducible realizations (representations up to a factor) of the maximal non-trivial central extension of the $(1+1)$ Galilei group, $\overline{G}(1+1)$, are obtained via the linear unitary irreducible representations of its maximal non-trivial central extension, $\overline{\overline{G}}(1+1)$. As an application we construct the Stratonovich–Weyl correspondence, which allows Moyal quantization of classical systems, for two cases of great physical interest: a system in a external variable force field and a variable-mass system.

1. Introduction

This paper is part of a wider programme started some years ago, which includes, as one of its goals, the construction of the unitary irreducible realizations (or representations up to a factor) of the kinematical groups of the space-time. Thus, we have constructed the locally operating representations of some of these groups—for instance: the Galilei $(3+1)$ group in [1]; the unitary irreducible realizations of the $(1+1)$ kinematical groups (Galilei, Poincaré and Newton–Hooke) in [2]; and the case of the Galilei $(2+1)$ group in [3]. Following this project, we present here the realizations of the maximal non-trivial central extension of the Galilei $(1+1)$ group, $\overline{G}(1+1)$. They have been obtained by means of obtaining linear unitary irreducible representations of a new group, $\overline{\overline{G}}(1+1)$, resulting from the central extension of $\overline{G}(1+1)$ by $U(1)$.

Unitary irreducible realizations (u.i.r.) can be applied to quantize, in the sense of Moyal [4], classical systems which have connected Lie groups as symmetry groups. This is carried out via the Stratonovich–Weyl (SW) correspondence (see [5]). This tool allows us to enlarge the phase-space formulation of quantum mechanics, given through the Moyal formalism [4], to physical systems with spin [6], relativistic systems [7] and systems in interaction [2, 3, 8] (see [9] for a review).

The construction of the SW correspondence requires the knowledge of the (classes of equivalence of) projective unitary irreducible representations (p.u.i.r.) of the symmetry group G as well as its coadjoint orbits. In addition we must put the two elements in correspondence. Note that (elementary) classical systems are associated with the coadjoint orbits of their corresponding symmetry groups. These orbits can be obtained taking into account the Kirillov–Kostant–Souriau theorem (see [10]), which establishes a local diffeomorphism between the homogeneous symplectic spaces of a connected Lie group G and the coadjoint orbits of its central extension by $U(1)$. On the other hand, the support

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space of a p.u.i.r. (u.i.r.) of the symmetry group is the quantum counterpart of a coadjoint orbit. Note that the correspondence between coadjoint orbits and p.u.i.r. (u.i.r.) for the groups involved in this paper is one to one—this is not the case in general.

The computation of the u.i.r. of G can be linearized by working with a new group, a central extension of the original one, whose linear unitary irreducible representations (l.u.i.r.) provide the origin for the u.i.r. of G. Moreover, this simplifies the research into the coadjoint orbits and the correspondence with the p.u.i.r. of G.

In [2] we constructed the SW kernels for physical systems associated with the one-dimensional kinematical groups: Galilei, Poincaré and Newton–Hooke. The central extensions of these groups are physically interpreted either as masses or as constant forces according to the group considered. So, we can introduce interactions in a system via central extensions of its symmetry group.

Recently, classical systems associated with a certain central extension of $\overline{G}(1+1)$ have been studied [11]. Thus, classical systems with non-constant acceleration (but whose derivative with respect to time—the jerk—is constant), can be considered. In terms of group theory, the force is no longer constant, because its corresponding infinitesimal generator now fails to be central and its commutator with the infinitesimal generator of the time translations provides the origin for a new central extension, directly related to the jerk. Systems with variable mass appear in a similar way.

In this paper we also study carefully the SW correspondence for some of these Galilean systems.

The paper is organized as follows: section 2 presents a short review of the SW correspondence. In section 3 we study the maximal non-trivial central extension of $\overline{G}(1+1)$, $\overline{\overline{G}}(1+1)$, and its coadjoint orbits. Section 4 deals with the problem of constructing the l.u.i.r. of $\overline{\overline{G}}(1+1)$. A complete list of representives for each equivalence class of l.u.i.r. is given. The procedure for obtaining the u.i.r of $\overline{G}(1+1)$ from the l.u.i.r. of $\overline{\overline{G}}(1+1)$ is also demonstrated. In section 5 we develop the SW correspondence for two particular elementary physical systems: (1) the case of variable force; and (2) the case of variable mass. We conclude with some remarks and an appendix on the group extensions, cohomology and linearization of projective unitary representations of Lie groups.

2. The Stratonovich–Weyl correspondence

The SW correspondence [5, 9] is a map that assigns linear operators on a Hilbert space to functions defined on a phase space. The SW correspondence profits from the existence of a (connected) Lie group, G, of the symmetry of the physical system under study. Thus, the phase space considered is a coadjoint orbit of this Lie group and the Hilbert space is the support space of a p.u.i.r. of G.

The first difficulty that arises is the construction of the p.u.i.r. of G; however, we are going to linearize the problem using a splitting group $\overline{G}$ of G, i.e., a Lie group $\overline{G}$ such that any p.u.i.r. of G can be lifted to a l.u.i.r. of $\overline{G}$ and, reciprocally, every l.u.i.r. of $\overline{G}$ provides a p.u.i.r. of G (see the appendix for more details).

A second one is that of assigning to each l.u.i.r. of $\overline{G}$ a coadjoint orbit of $\overline{G}$. This is carried out by the method of Kirillov [10] for constructing induced representations in the case of nilpotent groups.

The SW correspondence is performed via the existence of the SW kernel, $\mathbf{\Omega}$, which is a mapping transforming each point u of a given coadjoint orbit O of $\overline{G}$ into a selfadjoint operator, $\mathbf{\Omega}(u)$, on the Hilbert space $\mathcal{H}$ supporting the l.u.i.r. associated with the orbit. This mapping satisfies the following properties:

(1) $u \longmapsto \mathbf{\Omega}(u)$ is one to one;
(2) $\mathbf{\Omega}(u)$ is selfadjoint, $\forall u \in O$;
(3) $\mathrm{tr}[\mathbf{\Omega}(u)] = 1, \forall u \in O$—this trace is usually defined in a generalized sense;
(4) traciality:

$$\int_O \mathrm{tr}[\mathbf{\Omega}(u)\mathbf{\Omega}(v)]\mathbf{\Omega}(v)\ \mathrm{d}\mu(v) = \mathbf{\Omega}(u) \tag{2.1}$$

where μ is the G-invariant measure on O—this property means that $\mathrm{tr}[\mathbf{\Omega}(u)\mathbf{\Omega}(v)]$ behaves like a Dirac delta $\delta(u-v)$ with respect to the measure $\mu(v)$;

(5) covariance:

$$U(g)\mathbf{\Omega}(u)U(g^{-1}) = \mathbf{\Omega}(gu) \qquad \forall g \in \overline{G}; \forall u \in O \tag{2.2}$$

with $U(g)$ the l.u.i.r. of $\overline{G}$ associated with the orbit O and gu the transformed point of u produced by the coadjoint action of g.

Starting from a SW kernel for O, its associated SW correspondence is defined as follows. If $f(u)$ is a function on O, an operator A, on $\mathcal{H}$, is associated with it via

$$A = \int_O f(u)\mathbf{\Omega}(u)\ \mathrm{d}\mu(u). \tag{2.3}$$

The property of traciality allows us to obtain an inversion formula:

$$W_A(u) \equiv \mathrm{tr}[A\mathbf{\Omega}(u)] = \int_O f(v)\mathrm{tr}[\mathbf{\Omega}(u)\mathbf{\Omega}(v)]\ \mathrm{d}\mu(v) = f(u). \tag{2.4}$$

The function, $W_A(u)$, is usually called the Wigner function of A. Traciality also yields the following expression:

$$\mathrm{tr}[AB] = \int_O W_A(u)W_B(u)\ \mathrm{d}\mu(u) \tag{2.5}$$

allowing one to obtain quantum averages as in classical statistical mechanics.

The applicability of the SW correspondence is finally based on the construction of a non-commutative product—the so-called star or twisted product—for generalized functions on phase space equivalent to the product of operators on its corresponding Hilbert space. We can define this twisted product of two functions $f(u)$ and $g(u)$ on O as

$$(f * g)(u) = \int_O \int_O \mathrm{tr}[\mathbf{\Omega}(u)\mathbf{\Omega}(v)\mathbf{\Omega}(w)]f(v)g(w)\ \mathrm{d}\mu(v)\ \mathrm{d}\mu(w). \tag{2.6}$$

It is easy to verify that

$$(W_A * W_B)(u) = W_{AB}(u) \tag{2.7}$$

and

$$\int_O (f * g)(u)\ \mathrm{d}\mu(u) = \int_O f(u)g(u)\ \mathrm{d}\mu(u). \tag{2.8}$$

The term $\mathrm{tr}[\mathbf{\Omega}(u)\mathbf{\Omega}(v)\mathbf{\Omega}(w)]$ is called the tri-kernel of the SW correspondence. We will below give an explicit expression for it, together with the coadjoint orbit (phase space) and the SW kernel, completing the quantization for the cases under consideration.

The construction of a SW kernel requires the following steps: (1) choose an arbitrary point u_0 of O as origin; (2) produce an *ansatz* for a selfadjoint operator of trace one (with respect to a suitable trace) $\mathbf{\Omega}(u_0)$; (3) finally, define the kernel on the whole of O via

$$\mathbf{\Omega}(u) = \mathbf{\Omega}(gu_0) = U(g)\mathbf{\Omega}(u_0)U(g^{-1}) \tag{2.9}$$

where g is an element of $\overline{G}$ such that $gu_0 = u$. Note that this kernel is well defined if and only if

$$\mathbf{\Omega}(u_0) = U(\gamma)\mathbf{\Omega}(u_0)U(\gamma^{-1}) \qquad \forall \gamma \in \Gamma_{u_0} \tag{2.10}$$

where Γ_{u_0} is the isotopy group of u_0, i.e., $\Gamma_{u_0} = \{\gamma \in \overline{G} | \gamma u_0 = u_0\}$. This property, proved in [2], implies that $\mathbf{\Omega}(u)$, defined as above, is covariant. Reciprocally, if $\mathbf{\Omega}(u)$ is covariant, the property (2.10) holds. Remark that the covariance property guarantees that the SW kernel is well defined in the coadjoint orbit O—in other words, it is independent of the choice of section from O on $\overline{G}$.

It is interesting to include here a simple example in order to illustrate the meaning of the above construction for the standard quantum theory.

The main ingredient in the Weyl–Wigner–Moyal (or briefly Moyal) formulation of quantum mechanics [12, 13] is the twisted or Moyal product for functions on phase space. This product can be defined by using the Weyl mapping, i.e., a linear isomorphism between the space of the above-mentioned functions and the space of operators on a standard Hilbert space. The Weyl mapping can be introduced through the Grossmann–Royer operators [14, 15], which are defined as follows:

$$[\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})\varphi](\boldsymbol{x}) = 2^n \mathrm{e}^{2\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{q})}\, \varphi(2\boldsymbol{q} - \boldsymbol{x}) \tag{2.11}$$

where the standard $\mathbb{R}^{2n}$ phase space with coordinates $(\boldsymbol{q}, \boldsymbol{p})$ is assumed. These operators act as integral kernels in such a way that to a function f there corresponds the operator

$$W(f) = \frac{1}{2\pi} \int_{\mathbb{R}^{2n}} f(\boldsymbol{q}, \boldsymbol{p})\boldsymbol{K}(\boldsymbol{q}, \boldsymbol{p})\, \mathrm{d}\boldsymbol{q}\, \mathrm{d}\boldsymbol{p}. \tag{2.12}$$

The mapping is invertible, so the Moyal product can be defined by

$$f * g = W^{-1}(W(f)W(g)) \tag{2.13}$$

for which the explicit expression is

$$(f * g)(\boldsymbol{u}) = \frac{1}{\pi} \int_{\mathbb{R}^{4n}} f(\boldsymbol{v})g(\boldsymbol{w}) \exp[\mathrm{i}(\boldsymbol{u}J\boldsymbol{v} + \boldsymbol{v}J\boldsymbol{w} + \boldsymbol{w}J\boldsymbol{u})]\, \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{w} \tag{2.14}$$

where J is the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and where I_n and 0 are the n-dimensional identity and the $n \times n$ zero matrix, respectively, and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ stand for $(\boldsymbol{q}, \boldsymbol{p}), (\boldsymbol{q}', \boldsymbol{p}'), (\boldsymbol{q}'', \boldsymbol{p}'')$.

Now, we can construct the SW correspondence for the Heisenberg group H^{2n+1}, i.e., the set $\mathbb{R}^{2n+1}$ endowed with the following product:

$$(\boldsymbol{a}, \boldsymbol{b}, c)(\boldsymbol{a}', \boldsymbol{b}', c') = \left(\boldsymbol{a} + \boldsymbol{a}', \boldsymbol{b} + \boldsymbol{b}', c + c' + \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{b}' - \boldsymbol{a}' \cdot \boldsymbol{b})\right) \tag{2.15}$$

with $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^n$ and $c, c' \in \mathbb{R}$. The corresponding Lie algebra, $\mathcal{H}^{2n+1}$, is generated by the identity operator in $L^2(\mathbb{R}^n)$, I, and the n-dimensional position and momentum operators $\boldsymbol{Q}, \boldsymbol{P}$ with non-vanishing commutation relations $[Q_i, P_j] = \mathrm{i}I\delta_{ij}$. The coadjoint orbits of dimension greater than zero are all isomorphic to $\mathbb{R}^{2n}$ and yield the same kind of induced representations. A l.u.i.r. of H^{2n+1} associated with the coadjoint orbit specified by $z = 1$ is (let us write $(\boldsymbol{x}, \boldsymbol{y}, z)$ for the coordinates on a basis of $(\mathcal{H}^{2n+1})^*$ dual to the basis $\{Q_i, P_i, I\}$ of $\mathcal{H}^{2n+1}$):

$$[U(\boldsymbol{a}, \boldsymbol{b}, c)\varphi](\boldsymbol{\xi}) = \exp\left[-\mathrm{i}\left(c + \boldsymbol{b} \cdot \boldsymbol{\xi} + \frac{1}{2}\boldsymbol{a} \cdot \boldsymbol{b}\right)\right] \varphi(\boldsymbol{a} + \boldsymbol{\xi}) \tag{2.16}$$

where $\varphi \in L^2(\mathbb{R}^n)$.

Taking ($\boldsymbol{q} = \boldsymbol{x}/z$, $\boldsymbol{p} = \boldsymbol{y}$) as canonical coordinates on the orbit and choosing the point $(\mathbf{0}, \mathbf{0})$ as the origin (u_0) we can construct the *ansatz*

$$[\boldsymbol{\Omega}(u_0)\varphi](\boldsymbol{\xi}) = 2^n\,\varphi(-\boldsymbol{\xi}) \tag{2.17}$$

and obtain from it the SW kernel and tri-kernel which coincide with (2.11) and the integral kernel in (2.14), respectively.

3. The doubly extended Galilei (1 + 1) group

The Galilei group $G(1+1)$ is the group of transformations of the $(1+1)$ Newtonian space-time. Let (t, x) be the time and space coordinates of a point; the action of a generic element $g \equiv (b, a, v) = \mathrm{e}^{bH}\mathrm{e}^{aP}\mathrm{e}^{vK} \in G(1+1)$ on this point is

$$(t', x') = g(t, x) = (t + b, x + a + vt) \tag{3.1}$$

where b and a are the parameters of time and space translations and v corresponds to the Galilean inertial transformation. The Lie algebra of $G(1+1)$, $\mathcal{G}(1+1)$, is generated by H, P and K, which are the infinitesimal generators of time and space translations and Galilean inertial transformations, respectively. The Lie commutators of these generators are

$$[K, H] = P \qquad [K, P] = 0 \qquad [P, H] = 0. \tag{3.2}$$

Note that the Lie algebra just defined is isomorphic to that of the Heisenberg group H^{2+1}, which is itself a central extension of the group of translations of the phase plane.

3.1. *Central extension of $G(1+1)$*

The algebra $\mathcal{G}(1+1)$ admits a maximal non-trivial central extension by $\mathbb{R}^2$ [2, 16] (see also the appendix). Let $\overline{\mathcal{G}}(1+1)$ be the central extended algebra of $\mathcal{G}(1+1)$ with generators H, P, K, M and F, and non-vanishing commutation relations

$$[K, H] = P \qquad [K, P] = M \qquad [P, H] = F \tag{3.3}$$

where M and F are the two central generators linked with the extension. From a physical point of view M is related to the mass of an elementary physical system and F with a constant force field acting on this system, taking into account the theory of the p.u.i.r. of G [1, 17]. The group law for $\overline{G}(1+1)$ is

$$\overline{g}\,\overline{g}' = \left(\alpha + \alpha' + ab' + \frac{1}{2}vb'^2, \theta + \theta' + va' + \frac{1}{2}v^2b', b + b', a + a' + vb', v + v'\right) \tag{3.4}$$

with $\overline{g} = (\alpha, \theta, b, a, v) = \mathrm{e}^{\alpha F}\mathrm{e}^{\theta M}\mathrm{e}^{bH}\mathrm{e}^{aP}\mathrm{e}^{vK}$ [2]. The action on the space-time is like (3.1), i.e., $(t', x') = \overline{g}(t, x) = (t + b, x + a + vt)$, because the central generators act trivially on it.

3.2. *Central extension of $\overline{G}(1+1)$*

The algebra $\overline{\mathcal{G}}(1+1)$ admits a new maximal non-trivial central extension by $\mathbb{R}^3$, $\overline{\overline{\mathcal{G}}}(1+1)$, as is easy to prove by applying the cohomological methods shown in the appendix. This extended algebra is eight dimensional [11] and the non-zero Lie brackets in terms of the basis $\{H, P, K, M, F, R, D, S\}$ are as follows:

$$\begin{aligned} &[K, H] = P \qquad [K, M] = R \qquad [H, M] = -D \qquad [H, P] = -F \\ &[K, P] = M \qquad [K, F] = D \qquad [H, F] = -S. \end{aligned} \tag{3.5}$$

Note that M and F are no longer central generators, but this role is now played by R, D and S. The group law for $\overline{\overline{G}}(1+1)$ is given by

$$\begin{aligned}\overline{\overline{g}}\,\overline{\overline{g}}' = (&\rho + \rho' + v\theta' + \tfrac{1}{2}v^2a' + \tfrac{1}{6}v^3b', \delta + \delta' + v\alpha' - b\theta' - va'(b+b') - \tfrac{1}{4}v^2b'^2 \\ &-\tfrac{1}{2}bb'v^2, \sigma + \sigma' - b\alpha' - \tfrac{1}{2}ab'^2 - abb' - \tfrac{1}{3}vb'^3 - \tfrac{1}{2}vbb'^2, \alpha + \alpha' + ab' \\ &+\tfrac{1}{2}vb'^2, \theta + \theta' + va' + \tfrac{1}{2}v^2b', b + b', a + a' + vb', v + v') \qquad (3.6)\end{aligned}$$

where $\overline{\overline{g}} \equiv (\rho, \delta, \sigma, \alpha, \theta, b, a, v) = \mathrm{e}^{\rho R}\mathrm{e}^{\delta D}\mathrm{e}^{\sigma S}\mathrm{e}^{\alpha F}\mathrm{e}^{\theta M}\mathrm{e}^{bH}\mathrm{e}^{aP}\mathrm{e}^{vK} \in \overline{\overline{G}}(1+1)$. The inverse $\overline{\overline{g}}^{-1}$ of an element $\overline{\overline{g}}$ is

$$\begin{aligned}\overline{\overline{g}}^{-1} = (&-\rho + \theta v - \tfrac{1}{2}av^2 + \tfrac{1}{6}bv^3, -\delta - \theta b + \alpha v - \tfrac{1}{4}v^2b^2, -\sigma - \alpha b + \tfrac{1}{2}ab^2 - \tfrac{1}{3}vb^3, \\ &-\alpha + ab - \tfrac{1}{2}vb^2, -\theta + av - \tfrac{1}{2}bv^2, -b, -a + vb, -v). \qquad (3.7)\end{aligned}$$

3.3. Coadjoint orbits of $\overline{\overline{G}}(1+1)$

Let G be a Lie group, $\mathcal{G}$ its associated Lie algebra and $\mathcal{G}^*$ the dual space of $\mathcal{G}$. There exists an action of $\mathcal{G}$ on $\mathcal{G}$, called the adjoint action, defined by $\mathrm{ad}_X(Y) = [X, Y]$, $X, Y \in \mathcal{G}$. Exponentiation gives the adjoint action of the group G on its Lie algebra $\mathcal{G}$: $\mathrm{Ad}_{\mathrm{e}^X}(Y) = \mathrm{e}^{\mathrm{ad}_X}(Y)$, where $\mathrm{e}^X \in G$ and $X, Y \in \mathcal{G}$. The coadjoint action of G on $\mathcal{G}^*$ is given by

$$\langle \mathrm{coAd}_g\, a, X\rangle = \langle a, \mathrm{Ad}_{g^{-1}}X\rangle \qquad g \in G, X \in \mathcal{G}, a \in \mathcal{G}^* \qquad (3.8)$$

where $\langle\ ,\ \rangle$ denotes the product of $\mathcal{G}^*$ and $\mathcal{G}$. The coadjoint action of a generic element

$$g = (\rho, \delta, \sigma, \alpha, \theta, b, a, v) \in \overline{\overline{G}}(1+1)$$

on a point of $\overline{\overline{\mathcal{G}}}(1+1)^*$ of coordinates (r, d, s, f, m, h, p, k) in a basis dual to the basis $\{R, D, S, F, M, H, P, K\}$ of $\overline{\overline{\mathcal{G}}}(1+1)$ (and in this order) is

$$\begin{aligned}&r' = r \qquad d' = d \qquad s' = s \\ &f' = f - dv + sb \qquad m' = m - rv + db \\ &p' = p - mv + fb + \tfrac{1}{2}rv^2 - dvb + \tfrac{1}{2}sb^2 \qquad (3.9) \\ &k' = k + pb + m(a - bv) + \tfrac{1}{2}fb^2 + r(\theta + \tfrac{1}{2}bv^2 - av) + d(\alpha - \tfrac{1}{2}vb^2) + \tfrac{1}{6}sb^3 \\ &h' = h - pv + \tfrac{1}{2}mv^2 - fa - \tfrac{1}{6}rv^3 - d(\theta - av) - s\alpha.\end{aligned}$$

Note that this expression is equivalent to, but not identical, the one obtained in [11] because different, but equivalent, 2-cocycles are used.

For the sake of simplicity we will omit from now on bars denoting extensions over elements of a group determined when the group is arranged in advance.

The coadjoint orbits for this action are classified by displaying their invariants as follows (note that in this classification C_i denotes a constant).

(1) *The three extensions are non-vanishing.*

[1.1] The invariants characterizing this coadjoint orbit are: r, d, s, and

$$p - \frac{dmf}{d^2 - rs} + \frac{sm^2 + rf^2}{2(d^2 - rs)} = C_1.$$

This orbit is four dimensional (4D).

[1.2] Now the invariants are: $d, s, (r = d^2/s)$ and $m - (d/s)f = C_1$. It is also 4D.

[1.3] This orbit is 2D and is determined by $d, s, (r = d^2/s), (m = (d/s)f), p - f^2/2s = C_2$ and $k + (d/s)h - (C_2/s)f - f^3/6s^2 = C_3$.

(2) *Two non-vanishing extensions.*

[2.1] $(s = 0), r, d, p - mf/d + rf^2/2d^2 = C_1$.
[2.2] $(r = 0), d, s, p - mf/d + sm^2/2d^2 = C_1$.
[2.3] $(d = 0), r, s, p + sm^2 + rf^2/2rs = C_1$.

All of these orbits are 4D.

(3) *One non-vanishing extension.*

[3.1] $(s \neq 0), s, m$. It is 4D.
[3.2] $(s \neq 0), (m = 0), s, p - f^2/2s = C_1, k - fp/s + f^3/3s^2 = C_2$. It is 2D.
[3.3] $(r \neq 0), r, f$. This orbit is 4D.
[3.4] $(r \neq 0), (f = 0), r, p - m^2/2r = C_1, h - mp/r + m^3/3r^2 = C_2$. It is 2D.
[3.5] $(d \neq 0), d, p - mf/d = C_1$. This orbit is 4D.

(4) *The three extensions vanish* $(r = d = s = 0)$.

[4.1] $m, f, p^2 - 2mh - 2fk = C_1$.
[4.2] $(f = 0), p^2 - 2mh = C_1$.
[4.3] $(m = 0), p^2 - 2fk = C_1$.
[4.4] $(m = f = 0), (p \neq 0) p = C_1$.
[4.5] $(m = f = p = 0)$. Here the orbit is a point.

All of the remaining coadjoints orbits of this case (4) are 2D.

Now we analyse in more detail two of these sets of orbits because of the physical meaning, which we will see in section 5.

3.3.1. Variable force Let us consider case [3.1] of the above classification. The coadjoint action is given by

$$
\begin{aligned}
&s' = s \qquad f' = f + sb \qquad m' = m \\
&p' = p - mv + fb + \tfrac{1}{2} sb^2 \\
&k' = k + pb + m(a - bv) + \tfrac{1}{2} fb^2 + \tfrac{1}{6} sb^3 \\
&h' = h - pv + \tfrac{1}{2} mv^2 - fa - s\alpha
\end{aligned}
\tag{3.10}
$$

where

$$(0, 0, s', f', m', h', p', k')$$

is the point obtained by transformation of

$$(0, 0, s, f, m, h, p, k)$$

under the action of the element

$$g \equiv (\rho, \delta, \sigma, \alpha, \theta, b, a, v).$$

Note that expression (3.10) is obtained directly from (3.9) by making $r = d = 0$. In other words, we can consider in this case just the extension of $\overline{\mathcal{G}}(1+1)$ by $\langle S \rangle \equiv \mathbb{R}$.

The invariants characterizing the orbits are m and s or equivalently $m, j = s/m$ ($j \equiv$ jerk). These orbits will be denoted $O_{m,s} \equiv O_{m,j}$ and are 4D. A set of canonical coordinates $(\{q, p\} = 1,\ \{f, \phi\} = 1)$ for these orbits is given by

$$q = \frac{k}{m} \qquad p \qquad f \qquad \phi = \frac{1}{s}\left(h - \frac{p^2}{2m} + f\frac{k}{m}\right). \tag{3.11}$$

The symplectic 2-form is $\mathrm{d}\omega = \mathrm{d}q \wedge \mathrm{d}p + \mathrm{d}f \wedge \mathrm{d}\phi$. In these coordinates the coadjoint action (3.10) is rewritten as

$$\begin{aligned} q' &= q + \frac{p}{m}b + \frac{1}{2}\frac{f}{m}b^2 + \frac{1}{6}jb^3 + a - bv \\ p' &= p - mv + fb + \tfrac{1}{2}jmb^2 \qquad f' = f + jmb \\ \phi' &= \phi + (q+a)b + \frac{1}{2}\left(\frac{p}{m} - v\right)b^2 + \frac{1}{6}\frac{f}{m}b^3 + \frac{1}{24}jb^4 - \alpha. \end{aligned} \tag{3.12}$$

We can give a dynamical interpretation using the temporal evolution of the phase space coordinates, which is given, from (3.12), by

$$\begin{aligned} q(t) &= q(0) + \frac{p(0)}{m}t + \frac{f(0)}{2m}t^2 + \frac{1}{6}jt^3 \\ p(t) &= p(0) + f(0)t + \tfrac{1}{2}jmt^2 \qquad f(t) = f(0) + jmt \\ \phi(t) &= \phi(0) + q(0)t + \frac{1}{2}\frac{p(0)}{m}t^2 + \frac{1}{6}\frac{f(0)}{m}t^3 + \frac{1}{24}jt^4. \end{aligned} \tag{3.13}$$

Note that the temporal evolution of the position coordinate corresponds to a Galilean system with non-constant acceleration a but with $\mathrm{d}a/\mathrm{d}t = j$, a constant (the jerk). Taking into account the evolution equations of this system:

$$\frac{\mathrm{d}p}{\mathrm{d}b} = f \qquad \frac{\mathrm{d}f}{\mathrm{d}b} = mj \qquad \frac{\mathrm{d}q}{\mathrm{d}b} = \frac{p}{m} \qquad \frac{\mathrm{d}\phi}{\mathrm{d}b} = q \tag{3.14}$$

it is easy to find the corresponding Hamiltonian

$$H = \frac{p^2}{2m} - fq + jm\phi. \tag{3.15}$$

3.3.2. Variable mass Let us consider now the case [3.5]. The coadjoint action, taking $r = s = 0$ in (3.9), is

$$\begin{aligned} &d' = d \qquad f' = f - dv \qquad m' = m + db \\ &p' = p - mv + fb - dvb \qquad k' = k + pb + m(a - vb) + \tfrac{1}{2}fb^2 + d(\alpha - \tfrac{1}{2}vb^2) \\ &h' = h - pv + \tfrac{1}{2}mv^2 - fa - d(\theta - av). \end{aligned} \tag{3.16}$$

The invariants characterizing the orbits turn out to be the real parameters d and $c = p - mf/d$, so we have a stratum of 4D orbits $O_{d,c}$. For $O_{d,c}$

$$k \qquad \phi = \frac{f}{d} \qquad \mu = \frac{m}{d} \qquad \epsilon = h - \frac{fc}{d} - \frac{mf^2}{2d^2} \tag{3.17}$$

is a set of canonical coordinates, with a symplectic 2-form given by $\mathrm{d}\omega = \mathrm{d}k \wedge \mathrm{d}\phi + \mathrm{d}\mu \wedge d\epsilon$. The coadjoint action (3.16) in these new coordinates is

$$\begin{aligned} \mu' &= \mu + b \qquad \phi' = \phi - v \\ k' &= k + d\left(\alpha + a\mu + (\phi - v)\left(\mu b + \frac{b^2}{2}\right)\right) + cb \\ \epsilon' &= \epsilon - d\left(\theta + a(\phi - v) + \frac{b}{2}(\phi - v)^2\right). \end{aligned} \tag{3.18}$$

Once more, we can give a dynamical interpretation by deducing the laws of motion:

$$\begin{aligned} &\phi(t) = \phi(0) \qquad \mu(t) = \mu(0) + t \\ &k(t) = k(0) + d\phi(0)\left(\mu(0)t + \frac{t^2}{2}\right) + ct \\ &\epsilon(t) = \epsilon(0) - \frac{d}{2}\phi(0)^2 t \end{aligned} \tag{3.19}$$

and the corresponding evolution equations:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\mu}{\mathrm{d}t} = 1 \qquad \frac{\mathrm{d}k}{\mathrm{d}t} = c + d\mu\ \phi \qquad \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{d}{2}\phi^2 \tag{3.20}$$

associated with the Hamiltonian $H = \frac{1}{2}d\mu\phi^2 + c\phi + \epsilon$.

4. Linear unitary irreducible representations of the doubly extended Galilei (1 + 1) group

In order to construct the l.u.i.r. of $\overline{\overline{G}}(1+1)$ we are going to use the theory of Kirillov [10] because $\overline{\overline{G}}(1+1)$ is a nilpotent group of class four. First of all we present a short review of this theory.

Let us choose an element u in each coadjoint orbit; the subalgebra $\mathcal{H}$ of $\mathcal{G}$ such that $u([\mathcal{H},\mathcal{H}]) \equiv \langle u, [\mathcal{H},\mathcal{H}]\rangle = 0$ is said to be subordinate to u. The functional u allows the construction of a unitary one-dimensional representation Δ_u of the subgroup H associated with $\mathcal{H}$ via

$$\Delta_u(\mathrm{e}^h) = \mathrm{e}^{\mathrm{i}\langle u,h\rangle} \quad \forall h \in \mathcal{H}. \tag{4.1}$$

Starting from this representation Δ_u of H we induce a representation for the whole of G via

$$(\mathrm{D}(g)f)(x) = \Delta_u(\tau(x)^{-1}g\tau(g^{-1}x))f(g^{-1}x) \tag{4.2}$$

where τ is a normalized Borel section τ: $G/H \longrightarrow G$ ($\tau \circ \pi = \mathrm{id}_X$, with π the canonical projection π: $G \longrightarrow G/H$). If the measure on G/H is left G-invariant, the representation will be unitary.

Kirillov's theorem establishes the following.

(1) The representations of G induced by the representations Δ_u of H are irreducible if and only if the dimension of $\mathcal{H}$ is maximal among the subordinate subalgebras to u.

(2) Any irreducible representation of G can be obtained in this way up to equivalence.

(3) The representations induced by Δ_u and $\Delta_{u'}$ are equivalent if and only if u and u' belong to the same coadjoint orbit.

Note that orbits in the same stratum are diffeomorphic and their corresponding l.u.i.r. are formally similar but disequivalent. We are going to construct in a detailed manner the (equivalence classes of) the l.u.i.r. of $\overline{\overline{G}}(1+1)$ associated with the coadjoint orbits of variable force [3.1] and variable mass [3.5]. Thus, the reader will see how the method works. In addition, we will display the remaining (equivalence classes of) the l.u.i.r. of $\overline{\overline{G}}(1+1)$.

4.1. Variable force

Let us consider a point u of the orbit $O_{m,s}$ with coordinates $(s = mj, f = 0, m, h = 0, p = 0, k = 0)$. Two maximal subalgebras of $\overline{\overline{\mathcal{G}}}(1+1)$ subordinate to this point are $\langle R, D, S, F, M, P\rangle$ and $\langle R, D, S, F, M, K\rangle$. According with Kirillov's theory, both subordinate subalgebras induce equivalent representations. In the first case the configuration space of functions supporting the representation is of time–velocity kind while in the second one is of space-time kind. For methodological purposes, we are going to consider the second one ($\langle R, D, S, F, M, K\rangle$) but keeping in mind that for $\langle R, D, S, F, M, P\rangle$ the development is similar and the results are equivalent.

Let H be the group associated with the subalgebra $\mathcal{H} = \langle R, D, S, F, M, K\rangle$. A one-dimensional representation $\Delta_{m,s}$ of H is given by

$$\Delta_{m,s}(\mathrm{e}^h) = \mathrm{e}^{\langle u,h\rangle} = \mathrm{e}^{\mathrm{i}(\theta m + s\sigma)} \tag{4.3}$$

where $\mathrm{e}^h \equiv (\rho, \delta, \sigma, \alpha, \theta, 0, 0, v) \in H$ and $h = \rho R + (\delta + (1/2)\alpha v)D + \sigma S + \alpha F + \theta M + vK$.

The homogeneous space $\overline{\overline{G}}(1+1)/H = X$ is isomorphic to $\mathbb{R}^2$ and can be identified with the space-time manifold. A normalized Borel section τ: $X \longrightarrow \overline{\overline{G}}(1+1)$ $(\tau \circ \pi = \mathrm{id}_X$ with π the canonical projection π: $\overline{\overline{G}}(1+1) \longrightarrow \overline{\overline{G}}(1+1)/H)$, is defined by

$$\tau(t, x) := (0, 0, 0, 0, 0, t, x, 0). \tag{4.4}$$

The action of $\overline{\overline{G}}(1+1)$ on X is given by

$$g(t, x) := \pi(g\tau(t, x)) = (t + b, x + a + vt) \tag{4.5}$$

and the l.u.i.r. $D_{m,s}$ of $\overline{\overline{G}}(1+1)$ is defined by

$$[D_{m,s}(g)\varphi](g(t, x)) = \Delta_{m,s}(\tau^{-1}(g(t, x))g\tau(t, x))\varphi(t, x) \qquad \forall g \in \overline{\overline{G}}. \tag{4.6}$$

Since

$$\begin{aligned}\tau^{-1}&(g(t, x))g\tau(t, x)\\ &= (\rho + \tfrac{1}{2}v^2x + \tfrac{1}{6}v^3t, \delta + \theta(b + t) + \tfrac{1}{2}v^2t^2, \sigma + \alpha(t + b) + \tfrac{1}{2}at^2 + \tfrac{1}{6}vt^3,\\ &\quad \alpha + at + \tfrac{1}{2}vt^2, \theta + vx + \tfrac{1}{2}v^2t, 0, 0, v)\end{aligned} \tag{4.7}$$

we get that

$$[D_{m,s}(g)\varphi](g(t, x)) = \mathrm{e}^{\mathrm{i}m[\theta + vx + \frac{1}{2}v^2t]}\mathrm{e}^{\mathrm{i}s[\sigma + \alpha(t+b) + \frac{1}{2}at^2 + \frac{1}{6}vt^3]}\varphi(t, x) \tag{4.8}$$

or

$$\begin{aligned}[D_{m,s}&(g)\varphi](t, x)\\ &= \mathrm{e}^{\mathrm{i}m[\theta + v(x-a) + \frac{1}{2}v^2(b-t)]}\mathrm{e}^{\mathrm{i}s[\sigma + \alpha t + \frac{1}{2}a(t-b)^2 + \frac{1}{6}v(t-b)^3]}\varphi(t - b, x - a - v(t - b)).\end{aligned} \tag{4.9}$$

4.2. Variable mass

Let us consider the point u of the orbit $O_{d,c}$ with coordinates $(r = 0, d, s = 0, f = 0, m = 0, h = 0, p = c, k = 0)$ and take the maximal subalgebra subordinate to u, $\mathcal{H} = \langle R, D, S, F, M, P\rangle$. It is straightforward to construct a one-dimensional representation for the group H, associated with $\mathcal{H}$, given by

$$\Delta_{d,c}(\mathrm{e}^h) = \mathrm{e}^{\langle u,h\rangle} = \mathrm{e}^{\mathrm{i}(\delta d + ac)} \tag{4.10}$$

where $e^h = (\rho, \delta, \sigma, \alpha, \theta, 0, a, 0) \in H$ and $h = \rho R + \delta D + \sigma S + \alpha F + \theta M + aP$. The homogeneous space $\overline{\overline{G}}(1+1)/H = X$ is again isomorphic to $\mathbb{R}^2$ and can be identified with the time–velocity manifold. The expression for the normalized Borel section is in this case

$$\tau(t, w) := (0, 0, 0, 0, 0, t, 0, w) \tag{4.11}$$

and the action of $\overline{\overline{G}}(1+1)$ on X takes the form

$$g(t, w) := \pi(g\tau(t, w)) = (t + b, w + v). \tag{4.12}$$

The element for developing the induction process is

$$\begin{aligned}
\tau^{-1}&(g(t, w))g\tau(t, w) \\
&= (\rho + \tfrac{1}{6}v^3 t - (w + v)(\theta + \tfrac{1}{2}v^2 t) + \tfrac{1}{2}(w + v)^2(a + vt) \\
&\quad - \tfrac{1}{6}(w + v)^3(b + t), \delta + \theta(t + b) - \alpha(w + v) - \tfrac{1}{4}vt^2(v + 2w) - at(w + v), \\
&\quad \sigma + \tfrac{1}{2}at^2 + \tfrac{1}{6}vt^3 + \alpha(t - b), \alpha + at + \tfrac{1}{2}vt^2, \theta - a(w + v) \\
&\quad - \tfrac{1}{2}v^2 t - wvt, 0, a + vt, 0).
\end{aligned} \tag{4.13}$$

Finally, the l.u.i.r. is expressed as

$$[D_{d,c}(g)\varphi](t, w) = e^{ic[a+v(t-b)]}e^{id[\delta+\theta t-\alpha w-\frac{1}{4}v(2w-v)(t-b)^2-aw(t-b)]}\varphi(t - b, w - v). \tag{4.14}$$

4.3. The l.u.i.r of $\overline{\overline{G}}(1+1)$

We display in this section all of the (classes of equivalence) of the l.u.i.r. of $\overline{\overline{G}}(1+1)$. Each l.u.i.r., D, is labelled with a set of parameters which also characterizes a coadjoint orbit. So we follow the same order and enumeration of the coadjoint orbits as in section 3.1.

[1.1]

$$\begin{aligned}
[D_{r,d,s,C_1}&(g)\varphi](t, w) \\
&= e^{ir[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]} \\
&\quad \times e^{id[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}e^{is[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]} \\
&\quad \times e^{iC_1[a+(t-b)v]}\varphi(t - b, w - v).
\end{aligned}$$

[1.2]

$$\begin{aligned}
[D_{d,s,C_1}&(g)\varphi](t, w) \\
&= e^{i(d^2/s)[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]} \\
&\quad \times e^{id[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}e^{is[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]} \\
&\quad \times e^{iC_1[\theta+aw-vw(t-b)+\frac{1}{2}v^2(t-b)]}\varphi(t - b, w - v).
\end{aligned}$$

[1.3]

$$\begin{aligned}
[D_{d,s,C_2}&(g)\varphi](t, w) \\
&= e^{i(d^2/s)[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]} \\
&\quad \times e^{id[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}e^{is[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]} \\
&\quad \times e^{iC_2[a+(t-b)v]}\varphi(t - b, w - v).
\end{aligned}$$

For all three of these cases $\varphi \in \mathcal{L}^2(\mathbb{R}^2)$.

[2.1]

$$[D_{r,d,C_1}(g)\varphi](t,w) = \mathrm{e}^{\mathrm{i}r[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]} \times \mathrm{e}^{\mathrm{i}d[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}\mathrm{e}^{\mathrm{i}C_1[a+(t-b)v]}\varphi(t-b,w-v).$$

[2.2]

$$[D_{d,s,C_1}(g)\varphi](t,w) = \mathrm{e}^{\mathrm{i}d[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]} \times \mathrm{e}^{\mathrm{i}s[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]}\mathrm{e}^{\mathrm{i}C_1[a+(t-b)v]}\varphi(t-b,w-v).$$

[2.3]

$$[D_{r,s,C_1}(g)\varphi](t,w) = \mathrm{e}^{\mathrm{i}r[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]} \times \mathrm{e}^{\mathrm{i}s[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]}\mathrm{e}^{\mathrm{i}C_1[a+(t-b)v]}\varphi(t-b,w-v).$$

Also for all these three cases $\varphi \in \mathcal{L}^2(\mathbb{R}^2)$.

[3.1]

$$[D_{m,s}(g)\varphi](t,x) = \mathrm{e}^{\mathrm{i}m[\theta+v(x-a)+\frac{1}{2}v^2(b-t)]}\mathrm{e}^{\mathrm{i}s[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]} \times\varphi(t-b,x-a-v(t-b)).$$

[3.2]

$$[D_{s,C_1,C_2}(g)\varphi](t) = \mathrm{e}^{\mathrm{i}s[\sigma+\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]}\mathrm{e}^{\mathrm{i}C_1[a+(t-b)v]}\mathrm{e}^{\mathrm{i}C_2 v}\varphi(t-b).$$

[3.3]

$$[D_{r,f}(g)\varphi](t,w) = \mathrm{e}^{\mathrm{i}r[\rho+\frac{1}{2}a(4v^2+w^2)-\theta w+\frac{1}{2}(t-b)(vw^2-v^2w+\frac{1}{3}v^3)]}\mathrm{e}^{\mathrm{i}f[\alpha+a(t-b)+\frac{1}{2}v(t-b)^2]} \times\varphi(t-b,w-v).$$

[3.4]

$$[D_{r,C_1,C_2}(g)\varphi](w) = \mathrm{e}^{\mathrm{i}r[\rho-\theta w+\frac{1}{2}aw^2-\frac{1}{6}v^3)]}\mathrm{e}^{\mathrm{i}C_1[a-bw]}\mathrm{e}^{\mathrm{i}C_2 b}\varphi(w-v).$$

[3.5]

$$[D_{d,C_1}(g)\varphi](t,w) = \mathrm{e}^{\mathrm{i}d[\delta-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}\mathrm{e}^{\mathrm{i}C_1[a+(t-b)v]}\varphi(t-b,w-v).$$

For case [3.2] $\varphi \in \mathcal{L}^2(\mathbb{R})$; for the remaining ones $\varphi \in \mathcal{L}^2(\mathbb{R}^2)$.

[4.1]

$$[D_{f,m,C}(g)\varphi](x) = \mathrm{e}^{\mathrm{i}f[\alpha+\frac{1}{2}vb^2-b(x+a)]}\mathrm{e}^{\mathrm{i}m[\theta+\frac{1}{2}v^2b-va]}\mathrm{e}^{(\mathrm{i}/2f)[\hat{P}^2+2m\hat{H}+C]v}\varphi(x-a)$$

where $\hat{P} = -\mathrm{i}\partial_x$, $\hat{H} = -x$, $C = (l - C_1)$, $l \in \mathbb{R}$.

[4.2]

$$[D_{m,C_1}(g)\varphi](w) = \mathrm{e}^{\mathrm{i}m[\theta+\frac{1}{2}(w+v)^2b-(w+v)-bC_1/2m]}\varphi(w-v).$$

[4.3]

$$[D_{f,C_1}(g)\varphi](t) = \mathrm{e}^{\mathrm{i}f[\alpha+a(t-b)+\frac{1}{2}(t-b)^2-C_1v/f]}\varphi(t-b).$$

[4.4]

$[D_p(g)\varphi](t) = e^{ip[a+v(t-b)]}\varphi(t-b).$

Also for all four of these cases $\varphi \in \mathcal{L}^2(\mathbb{R})$.

[4.5] The representations are one dimensional.

4.4. Unitary irreducible realizations of $\overline{G}(1+1)$

Once one has the l.u.i.r. of $\overline{\overline{G}}(1+1)$, the u.i.r. of $\overline{G}(1+1)$ can be obtained as follows (see the appendix). Let χ be a normalized Borel section from $\overline{G}(1+1)$ on $\overline{\overline{G}}(1+1)$ ($\chi \circ p = \mathrm{id}_{\overline{G}(1+1)}$). Since all the l.u.i.r. of $\overline{\overline{G}}(1+1)$ displayed above, restricted to $\hat{H}^2(G, U(1))$, belong to $U(1)$ we get the u.i.r. $\mathcal{U}$ of $\overline{G}(1+1)$ in the following way:

$$\mathcal{U}(\overline{g}) = (D \circ \chi)(\overline{g}) = D(\chi(\overline{g})) \qquad \overline{g} \in \overline{G}(1+1). \tag{4.15}$$

A suitable choice for χ: $\overline{G}(1+1) \to \overline{\overline{G}}(1+1)$ is

$$\chi(\overline{g}) \equiv \chi((\theta, \alpha, b, a, v)) = (0, 0, 0, \theta, \alpha, b, a, v). \tag{4.16}$$

For instance, the u.i.r. associated with the l.u.i.r. [3.1] (variable force) and [3.5] (variable mass) are, respectively:

$$\begin{aligned}\mathcal{U}_{m,s}(\overline{g}) &= [D_{m,s}(\chi(\overline{g}))\varphi](t, x)\\ &= e^{im[\theta+v(x-a)+\frac{1}{2}v^2(b-t)]}e^{is[\alpha t+\frac{1}{2}a(t-b)^2+\frac{1}{6}v(t-b)^3]}\varphi(t-b, x-a-v(t-b))\end{aligned} \tag{4.17}$$

$$\begin{aligned}\mathcal{U}_{d,C_1}(\overline{g}) &= [D_{d,C_1}(\chi(\overline{g}))\varphi](t, w)\\ &= e^{id[-\alpha w+\theta t-aw(t-b)-\frac{1}{4}v(t-b)^2(2w-v)]}e^{iC_1[a+(t-b)v]}\varphi(t-b, w-v).\end{aligned} \tag{4.18}$$

5. Stratonovich–Weyl kernels for variable force and variable mass

This section is devoted to the construction of the SW kernels for some physical systems whose phase spaces are some of the coadjoint orbits studied before. More specifically, we shall consider the cases of a classical system interacting with a variable force and a system of variable mass, both cases having the extended Galilei group, $\overline{G}(1+1)$, as the symmetry group. We have presented the method for constructing these SW kernels in section 2 and we are going to apply it here.

5.1. Variable force

Let us consider the point $\mathbf{0} \equiv (q = 0, p = 0, f = 0, \phi = 0)$ as the origin of the coadjoint orbit $O_{m,s}$. The point $u = (q, p, f, \phi)$ can be considered as the origin transformed by the group element $g \equiv (0, 0, 0, \alpha, 0, b, a, v)$ of $\overline{\overline{G}}(1+1)$ such that

$$\begin{aligned}\alpha &= -\phi + \frac{f}{s}q + \frac{f^4}{8ms^3} - \frac{pf^2}{2ms^2} \qquad & b &= \frac{f}{s}\\ a &= q + \frac{f^3}{3ms^2} - \frac{pf}{ms} \qquad & v &= \frac{f^2}{2ms} - \frac{p}{m}.\end{aligned} \tag{5.1}$$

The isotopy group of the origin, $\overline{\overline{\Gamma}}$, is

$$\overline{\overline{\Gamma}} \equiv \{\gamma \in \overline{\overline{G}} | \gamma = (\rho, \delta, \sigma, 0, \theta, 0, 0, 0)\}. \tag{5.2}$$

The l.u.i.r. of $\overline{\overline{G}}$ restricted to $\overline{\overline{\Gamma}}$ are given by

$$[D_{m,s}(\gamma)\varphi](t,x) = \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{\mathrm{i}s\sigma}\varphi(t,x) \qquad (m,s)\in\mathbb{R}^2. \tag{5.3}$$

The SW kernel can be defined at the origin as follows:

$$[\Omega(\mathbf{0})\varphi](t,x) := 2^2\varphi(-t,-x). \tag{5.4}$$

This kernel is covariant as can be proved by checking that $[D(\gamma),\Omega(\mathbf{0})]=0$, $\gamma\in\overline{\overline{\Gamma}}$. The value of the SW kernel in any point $u=(q,p,f,\phi)$ of the orbit is

$$\begin{aligned}
&[\Omega(u)\varphi](t,x)\\
&\qquad = [D(g)\Omega(\mathbf{0})D(g)^{-1}\varphi](t,x)\\
&\qquad = 2^2\mathrm{e}^{\mathrm{i}m[2v(a-x)+v^2(b-t)]}\mathrm{e}^{\mathrm{i}s[2\alpha(t-b)+\frac{1}{3}v(t^3-b^3)+vbt(b-t)]}\varphi(2b-t,2a-x)
\end{aligned}\tag{5.5}$$

with $g\equiv(0,0,0,\alpha,0,b,a,v)$ given by (5.1).

It is straightforward to prove that $\mathrm{tr}[\Omega(q,p,f,\phi)]=1$. The property of traciality (2.1) can be checked using the equivalent statement (see [3]) $\mathrm{tr}[\Omega(\mathbf{0})\Omega(u)]=\delta(u)$, $\forall u\in O_{m,s}$:

$$\begin{aligned}
&\mathrm{tr}[\Omega(\mathbf{0})\Omega(q,p,f,\phi)]\\
&\qquad = \int_{R^2}\langle t,x|\Omega(\mathbf{0})\Omega(q,p,f,\phi)|t,x\rangle\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad = 2^2\int_{R^2}\langle t,x|\Omega(q,p,f,\phi)|-t,-x\rangle\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad = 2^2\sigma\,\delta(f)\int_{R^2}\exp\left\{-\mathrm{i}m\left[2\frac{-p}{m}(q+x)+\frac{p^2}{m^2}t\right]\right\}\\
&\qquad\quad \times\exp\left\{-\mathrm{i}\sigma\left[2\phi t+\frac{pt^3}{3m}\right]\right\}\delta(q)\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad = 2^2\sigma\,\delta(f)\,\delta(q)\int_{R^2}\mathrm{e}^{2\mathrm{i}px}\mathrm{e}^{-2\mathrm{i}[p^2/2m+\sigma\phi+pt^2/6m]t}\,\mathrm{d}t\,\mathrm{d}x\\
&\qquad = 2\sigma\,\delta(f)\,\delta(q)\,\delta(p)\int_R\mathrm{e}^{-2\mathrm{i}\sigma\phi t}\,\mathrm{d}t\\
&\qquad = \delta(f)\,\delta(p)\,\delta(q)\,\delta(\phi).
\end{aligned}\tag{5.6}$$

On the other hand we have computed the tri-kernel (2.6) obtaining that

$$\begin{aligned}
&\mathrm{tr}[\Omega(q,p,f,\phi)\Omega(q',p',f',\phi')\Omega(q'',p'',f'',\phi'')]\\
&\qquad = 2^4\exp\{-\mathrm{i}m[2v(a'-a'')+2v'(a''-a)+2v''(a-a')\\
&\qquad\quad +v^2(b'-b'')+v'^2(b''-b)+v''^2(b-b')]\}\\
&\qquad\quad \times\exp\{-\mathrm{i}\sigma[(2\alpha-vb(b''-b'+b))(b''-b')\\
&\qquad\quad +(2\alpha'-v'b'(-b''+b'+b))(b-b'')\\
&\qquad\quad +(2\alpha''-v''b''(b''+b'-b))(b'-b)+\tfrac{1}{3}v((b''-b'+b)^3-b^3)\\
&\qquad\quad +\tfrac{1}{3}v'((-b''+b'+b)^3-b'^3)+\tfrac{1}{3}v''((b''+b'-b)^3-b''^3)]\}.
\end{aligned}\tag{5.7}$$

For the sake of simplicity we have preferred to write the tri-kernel in terms of the group element $g=(0,0,0,\alpha,0,b,a,v)$, which according to (5.1) appears as an implicit function of the canonical coordinates of the orbit. This preserves the symmetric aspect of the formula, which resembles that of the tri-kernel for the Heisenberg group [2].

5.2. Variable mass

In this case we have chosen as the origin of the orbit the point of coordinates $\mathbf{0} \equiv (k = 0, \phi = 0, \mu = 0, \epsilon = 0)$. By the action of the group element

$$g \equiv \left(0, \frac{k - c\mu}{\delta} - \frac{1}{2}\phi\mu^2, \frac{-\epsilon}{\delta} - \frac{1}{2}\mu\phi^2, \mu, 0, -\phi\right) \tag{5.8}$$

we transform the origin into the point $u = (k, \phi, \mu, \epsilon)$ of the orbit $O_{\delta,c}$. The isotopy group of the origin is

$$\overline{\overline{\Gamma}} \equiv \{\gamma \in \overline{\overline{G}} | \gamma = (\rho, \delta, \sigma, 0, 0, 0, a, 0)\}$$

and the restriction of the representation $D_{\delta,c}$ of $\overline{\overline{G}}$ to $\overline{\overline{\Gamma}}$ is

$$[D_{d,c}(\gamma)\varphi](t, w) = \mathrm{e}^{\mathrm{i}ca}\mathrm{e}^{\mathrm{i}\delta d}\varphi(t, w). \tag{5.9}$$

Taking as the *ansatz* for the value of the SW kernel at the origin

$$[\Omega(\mathbf{0})\varphi](t, w) := 2^2\varphi(-t, -w) \tag{5.10}$$

and since $[D(\gamma), \Omega(\mathbf{0})] = 0$ we can define the SW kernel at any point of the orbit using the covariance property (2.2); we obtain that

$$\begin{aligned}
&[\Omega(k, \phi, \mu, \epsilon)\varphi](t, w) \\
&\quad = [D(g)\Omega(\mathbf{0})D(g)^{-1}\varphi](t, w) \\
&\quad = 2^2\mathrm{e}^{\mathrm{i}d[(-2\epsilon/d-\mu\phi^2)(t-\mu)-(2(k-c\mu)/d-\phi\mu^2)(\phi+w)+\phi(\phi+w)(t-\mu)^2]} \\
&\qquad \times\mathrm{e}^{-2\mathrm{i}c\phi(t-\mu)}\varphi(2\mu - t, -2\phi - w)
\end{aligned} \tag{5.11}$$

with g given by (5.8).

We can prove the traciality property and compute the tri-kernel. We get, respectively, that

$$\begin{aligned}
&\mathrm{tr}[\Omega(\mathbf{0})\Omega(k, \phi, \mu, \epsilon)] \\
&\quad = \int_{R^2} \langle t, w|\Omega(\mathbf{0})\Omega(k, \phi, \mu, \epsilon)|t, w\rangle\, \mathrm{d}t\, \mathrm{d}w \\
&\quad = 2^2 \int_{R^2} \langle t, w|\Omega(k, \phi, \mu, \epsilon)| - t, -w\rangle\, \mathrm{d}t\, \mathrm{d}w \\
&\quad = 2^4 \int_{R^2} \mathrm{e}^{-\mathrm{i}\delta[(-2\epsilon/\delta-\mu\phi^2/2)(-t-\mu)-(2(k-c\mu)/\delta-\phi\mu^2)(\phi-w)+\phi(\phi-w)(-t-\mu)^2]} \\
&\qquad \times\mathrm{e}^{2\mathrm{i}c\phi(-t-\mu)}\langle t, w|2\mu + t, -2\phi + w\rangle\, \mathrm{d}t\, \mathrm{d}w \\
&\quad = 2^2\, \delta(\mu)\, \delta(\phi) \int_{R^2} \mathrm{e}^{-2\mathrm{i}\epsilon t}\mathrm{e}^{-2\mathrm{i}kw}\, \mathrm{d}t\, \mathrm{d}w \\
&\quad = \delta(k)\, \delta(\phi)\, \delta(\mu)\, \delta(\epsilon)
\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}
&\mathrm{tr}[\Omega(k, \phi, \mu, \epsilon)\Omega(k', \phi', \mu', \epsilon')\Omega(k'', \phi'', \mu'', \epsilon'')] \\
&\quad = 2^4 \exp\{2\mathrm{i}c[\phi(\mu'' - \mu') + \phi'(\mu - \mu'') + \phi''(\mu' - \mu)]\} \\
&\qquad \times \exp\{-\mathrm{i}\delta[(-2/\delta)(\epsilon(\mu'' - \mu') + \epsilon'(\mu - \mu'') + \epsilon''(\mu' - \mu)) \\
&\qquad +2\alpha(\phi'' - \phi') + 2\alpha'(\phi - \phi'') + 2\alpha''(\phi' - \phi) + \phi(\phi' - \phi'')(\mu'' - \mu')^2 \\
&\qquad +\phi'(\phi'' - \phi)(\mu - \mu'')^2 + \phi''(\phi - \phi')(\mu' - \mu)^2]\}
\end{aligned} \tag{5.13}$$

where $\alpha = (k - c\mu)/\delta - \frac{1}{2}\phi\mu^2$ and the remarks made regarding (5.7) remain also valid here.

We should finish this section by noting that the tri-kernels just calculated permit the definition of twisted products, which provide the basic tool for the Moyal formulation of quantum mechanics.

6. Conclusions

As we have mentioned above, central extensions of the symmetry groups of physical systems provide us with the opportunity of introducing interactions in a simple way. Remember that the central extension associated with the commutator $[P, H] = F$ in the Galilean (1+1) case is physically interpreted as a constant force. Despite these expectations it was impossible to obtain the SW correspondence in some cases—for instance when mass and force extensions are simultaneously considered in $\overline{G}(1+1)$ [2]. Now, in this paper we have obtained a SW kernel for a massive one-dimensional Galilean system interacting with a non-constant force.

It is worth mentioning that for the remaining $(1+1)$ kinematical groups (Poincaré and Newton–Hooke) there is no double extension, i.e., the central extension of their first central extension is trivial [2].

Research on the SW correspondence for $(2+1)$ Poincaré and Newton–Hooke systems is in progress.

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Appendix

A.1. Central extensions of groups and cohomology

A group E is a central extension of the group G by the Abelian group A if the following sequence is exact:

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1.$$

It is well known that research on the central extensions, up to equivalence, forms a cohomological problem [18, 19] in the sense that an extension of G by A has associated with it an action $\Theta\colon G \to \operatorname{Aut} A$ and an element $[w] \in H^2_\Theta(G, A)$. So, there exists a bijection between the set of classes of central extensions of G by A, $\operatorname{Ext}_\Theta(G, A)$, and the second cohomology group of G, $H^2_\Theta(G, A)$. We will not discuss here the topological details related to this problem.

On the other hand, the construction of the central extensions of a Lie group G by another (Abelian) Lie group A is equivalent (at least locally) to finding the central extensions of the Lie algebra of G, $\mathcal{G}$, by the Lie algebra, $\mathcal{A}$, of A. For the cases studied in this paper, the groups are connected and simply connected so $H^2(G, A) = H^2(\mathcal{G}, \mathcal{A})$.

We present now a short review on the cohomology of Lie algebras [19].

Let us consider a Lie algebra $\mathcal{G}$ and a $\mathcal{G}$-module $\mathcal{A}$, i.e. $\mathcal{A}$ is a linear space supporting a linear representation ψ of $\mathcal{G}$ which satisfies

$$\psi(X)\psi(Y) - \psi(Y)\psi(X) = \psi([X, Y]). \tag{A.1}$$

An n-cochain is an n-linear alternating mapping $\omega_n\colon \mathcal{G}\times\mathcal{G}\times\overset{n\ \text{times}}{\cdots}\times\mathcal{G}\to\mathcal{A}$. The space of n-cochains is denoted by $C^n(\mathcal{G},\mathcal{A})$. For every $n\in N$ there exists a linear map $\delta_n\colon C^n(\mathcal{G},\mathcal{A})\longrightarrow C^{n+1}(\mathcal{G},\mathcal{A})$ defined by

$$
\begin{aligned}
(\delta_n\omega)&(X_1,\ldots,X_{n+1})\\
&=\sum_{i=1}^{n+1}(-1)^{i+1}\psi(X_i)\omega(X_1,\ldots,X_{i-1},\hat{X}_i,X_{i+1},\ldots,X_{n+1})\\
&\quad+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{n+1})
\end{aligned}
\tag{A.2}
$$

where the hat of $\hat{X}$ indicates that X is omitted. It is possible to prove that $\delta_n\circ\delta_n=0,\ \forall n$. The operator δ, called the coboundary operator, is defined on $C(\mathcal{G},\mathcal{A})=\bigoplus_{n=0}^{\infty}C^n(\mathcal{G},\mathcal{A})$ in such a way that $\delta|_{C^n}=\delta_n$ and satisfies $\delta^2=0$.

In $C^n(\mathcal{G},\mathcal{A})$ we can consider the following subsets:

$$B^n(\mathcal{G},\mathcal{A})=\{\omega\in C^n|\exists\,\alpha\in C^{n-1}\text{ such that }\omega=\delta\alpha\}$$

$$Z^n(\mathcal{G},\mathcal{A})=\{\omega\in C^n|\delta\omega=0\}.$$

It is obvious that $B^n\subset Z^n$. The elements of B^n and Z^n are called n-coboundaries and n-cocycles, respectively. The n-cohomology group $H^n(\mathcal{G},\mathcal{A})$ is defined by $H^n=Z^n/B^n$.

In our particular case the $\mathcal{G}$-module is $\mathbb{R}$ and the representation $\psi=0$. The space $C^n(\mathcal{G},\mathbb{R})$ can be identified with

$$\mathcal{G}^*\wedge\overset{n\ \text{times}}{\cdots}\wedge\mathcal{G}^*$$

where $\mathcal{G}^*$ is the vector space that is the dual of $\mathcal{G}$. The 2-cocycles ω are characterized by the property

$$\omega([X_1,X_2],X_3)+\omega([X_3,X_1],X_2)+\omega([X_2,X_3],X_1)=0 \tag{A.3}$$

and the 2-coboundaries by

$$\omega(X_1,X_2)=-\alpha([X_1,X_2]) \tag{A.4}$$

with α a 1-cocycle.

As an example, the computation of the second cohomology group of $\mathcal{G}(1+1)$ is performed as follows: let us consider a 2-cochain as an element of $\mathcal{G}(1+1)^*\wedge\mathcal{G}(1+1)^*$, i.e, $\omega=mK^*\wedge P^*+fP^*\wedge H^*+aH^*\wedge K^*$, where $\{H^*,P^*,K^*\}$ constitutes a basis of $\mathcal{G}(1+1)^*$, which is the dual of the basis $\{H,P,K\}$ of $\mathcal{G}(1+1)$, and $m,f,a\in\mathbb{R}$. The condition for the 2-cocycle (A.3) is trivially satisfied by H, P and K with the commutation rules (3.2). However, the condition for the 2-coboundary (A.4) implies that the 2-cocycle $H^*\wedge K^*$ is trivial, i.e., a 2-coboundary. Hence,

$$H^2(\mathcal{G}(1+1),\mathbb{R})=\{[m,f]/m,f\in\mathbb{R}\}\equiv\mathbb{R}^2. \tag{A.5}$$

The two non-trivial 2-cocycles $K^*\wedge P^*$ and $P^*\wedge H^*$ are linked with the new Lie commutators $[K,P]=mI$ and $[P,H]=fI$, respectively (see (3.3)).

For the two groups $G(1+1)$ and $\overline{G}(1+1)$ involved in this paper, $H^2(\mathcal{G},\mathbb{R})=H^2(G,U(1))$. A 2-cocycle of the class $[m,f]\in H^2(G(1+1),U(1))$ is obtained by integrating the new commutation rules, which correspond to the extended algebra. In other words, we compute the group law for the extended group (see (3.4)). So,

$$
\begin{aligned}
W_{m,f}(g,g')&=\exp\left\{\mathrm{i}m\left(ab'+\frac{1}{2}vb'^2\right)\right\}\\
&\quad\times\exp\left\{\mathrm{i}f\left(va'+\frac{1}{2}v^2b'\right)\right\}\qquad g,g'\in G(1+1).
\end{aligned}
\tag{A.6}
$$

Similarly for an element $[r, d, s, m, f] \in H^2(\overline{G}(1+1), U(1))$ we find the following lifting:

$$
\begin{aligned}
W_{r,d,s,m,f}(\overline{g}, \overline{g}')& \\
= {}& \exp\{\mathrm{i}r(v\theta' + \tfrac{1}{2}v^2 a' + \tfrac{1}{6}v^3 b')\} \\
&\times \exp\{\mathrm{i}d(v\alpha' - b\theta' - va'(b+b') - \tfrac{1}{4}v^2 b'^2 - \tfrac{1}{2}bb'v^2)\} \\
&\times \exp\{\mathrm{i}s(-b\alpha' - \tfrac{1}{2}ab'^2 - abb' - \tfrac{1}{3}vb'^3 - \tfrac{1}{2}vbb'^2)\} \\
&\times \exp\{\mathrm{i}m(-\tfrac{1}{2}bb'v^2 ab' + \tfrac{1}{2}vb'^2)\} \exp\{\mathrm{i}f(va' + \tfrac{1}{2}v^2 b')\} \qquad \text{(A.7)}
\end{aligned}
$$

with $\overline{g}, \overline{g}' \in \overline{G}(1+1)$. Note that in (3.4) and (3.6) we have considered representatives of the cohomology classes [1, 1] and [1, 1, 1, 1, 1], respectively.

A.2. Linearization of projective representations of Lie groups

As is well known, the projective unitary irreducible representation (p.u.i.r.) of a group G is a homomorphism, P, of G on $PU(\mathcal{H})$, the group of projective unitary operators on the Hilbert space $\mathcal{H}$. Moreover $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$, where $U(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ and $U(1)$ the group of operators of the form $\alpha\mathbf{I}, \alpha \in \mathbb{C}, |\alpha|^2 = 1$. In other words, we have the exact sequence

$$1 \rightarrow U(1) \xrightarrow{i} U(\mathcal{H}) \xrightarrow{\pi} PU(\mathcal{H}) \rightarrow 1.$$

When we say that $\mathcal{U}$ is a unitary representation up to a factor or a realization (u.r.) we are usually making reference to a mapping of G on $U(\mathcal{H})$ such that $\pi \circ U$ is a p.u.r. of G. So, if U is a unitary realization of G

$$U(g')U(g) = \xi(g', g)U(g'g) \qquad \forall g \in G$$

where $\xi\colon G \times G \rightarrow U(1)$ is called a factor system of G. The associativity of U imposes the following condition on ξ:

$$\xi(g_1, g_2)\xi(g_1 g_2, g_3) = \xi(g_2, g_3)\xi(g_1, g_2 g_3)$$

i.e., ξ is a 2-cocycle ($\xi \in \mathrm{Z}^2(G, U(1))$).

The unitary equivalence of two u.i.r. of G, U and U', is defined by $U'(g) = \lambda(g)TU(g)T^{-1}$ ($\forall g \in G$), with T a unitary operator and for a map $\lambda\colon G \longrightarrow U(1)$. This implies that their corresponding factor systems differ in a 2-coboundary ($\delta\lambda$) as is easy to prove. So, the classes (of equivalence) of the p.u.i.r. of G are in one-to-one correspondence with the elements of $H^2(G, U(1))$.

The pair $(\tilde{G}, p)$ is said to be a splitting group [20] of a Lie group G if $p\colon \tilde{G} \rightarrow G$ is an epimorphism and any p.u.i.r of G can be lifted to a l.u.i.r. of $\tilde{G}$ mapping $\mathrm{Ker}\, p = A$ to $U(1)$. On the other hand, a l.u.i.r. D of $\tilde{G}$ is said to be A-split if $D(A) \subset U(1)$. These representations of $\tilde{G}$ produce by quotient the p.u.i.r. of G.

In the following we will only consider central extensions of G by an Abelian group as candidates for splitting groups of G. It can be proved that for each normalized Borel section $\chi\colon G \rightarrow \tilde{G}$ ($p \circ \chi = \mathrm{id}_G$) and for each A-split l.u.r. of $\tilde{G}$ there exists a u.r. $\mathcal{U}$ of G given by $\mathcal{U} = D \circ \chi$, whose factor system is $\xi = D \circ W_\chi$, where W_χ is the 2-cocycle associated with the central extension

$$1 \rightarrow A \xrightarrow{i} \tilde{G} \xrightarrow{p} G \rightarrow 1$$

defined by χ, i.e., $W_\chi(g,g') = \chi(g)\chi(g')(\chi(gg'))^{-1}$. This can be illustrated by the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \stackrel{i}{\longrightarrow} & U(\mathcal{H}) & \stackrel{\pi}{\longrightarrow} & PU(\mathcal{H}) & \longrightarrow & 1 \\ & & \uparrow D|_A & & \uparrow D & \nwarrow \mathcal{U} & \uparrow P & & \\ 1 & \longrightarrow & A & \stackrel{i}{\longrightarrow} & \tilde{G} & \stackrel{p}{\longrightarrow} & G & \longrightarrow & 1 \end{array}$$

A 'minimal' splitting group $\tilde{G}$ of G or representation group is a central extension of G by the dual of $H^2(G,U(1))$, $\hat{H}^2(G,U(1))$, such that $\hat{H}^2(G,U(1))$ is contained in the derived group of $\tilde{G}$ [17]. Remember that the dual of an Abelian group is the group of its u.i. representations.

In order to show how the theory works, let us construct that 'minimal' splitting group for $G(1+1)$. The non-vanishing Lie commutators of the central extension of $\mathcal{G}(1+1)$ are

$$[K,H] = P \qquad [K,P] = mI \qquad [P,H] = fI \qquad m, f \in \mathbb{R}.$$

The second cohomology group of $G(1+1)$, $H^2(G,U(1))$, and its dual $\hat{H}^2(G,U(1))$ are both isomorphic to $\mathbb{R}^2$. Note that $[mf] \longrightarrow \mathrm{e}^{\mathrm{i}(\alpha f+\theta m)} \in U(1)$, $(\alpha,\theta) \in \mathbb{R}^2$. Fixing the values of $m = 1$ and $f = 1$, we use the notation $mI = M$ and $fI = F$, and 'integrating' the Lie commutators of the Lie algebra generated by M, F, H, P and K we get a group with the law given by (3.4). A lifting of $[m, f] \in H^2(G,U(1))$ is given by (A.6).

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